<u>Optimization</u>

<u>Last time</u>: gradient = direction of maximal increase in f.

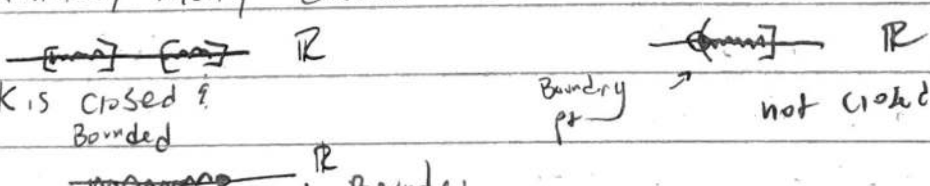
<u>A critical pt (CP)</u> of f is a point $\vec{p}$ in the domain of f where either ∇f @ $\vec{p} = 0$ or $\nabla f(\vec{p})$ does not exist

<u>prop (Fermat extremum thm)</u>: If f has a local extreme value @ p, then $\vec{p}$ is a crit. pt. of f

to do local optimization we also need:

<u>Prop (Extreme Value Theorem)</u>: if f is defined on a <u>closed</u> & <u>bounded</u> subset $K \subseteq \mathbb{R}^n$, then f attains its global extrema (on K)

<u>What is "closed & bounded"?</u>: In $\mathbb{R}^1$ a set is closed & bounded iff it is a union of finitely many closed and bounded intervals



$\mathbb{R}$ $\mathbb{R}$

K is closed & Bounded

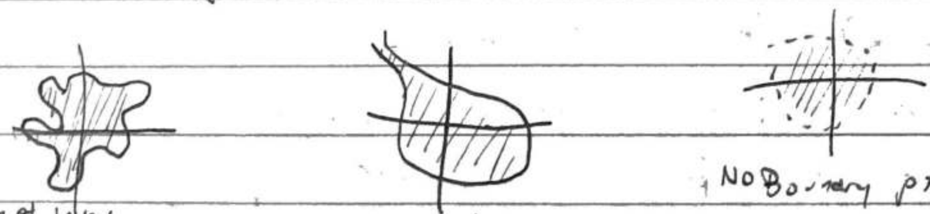
Boundary pt

not closed!

$\mathbb{R}$

x not Bounded

In $\mathbb{R}^2$;



All Boundary pt belong to set

K is closed & Bounded

Nothing goes to infinity

X is not Bounded

No Boundary pts belong

<u>Note</u>! A set is closed & Bounded when it is bounded and its "separating points from the rest of $\mathbb{R}^n$" are all in the set →

This suggests a method for optimizing global values on a closed and bounded subset

Alg (compact set method): Let f be a function defined on a closed & bounded subset K. To compute global extrema of f on K:

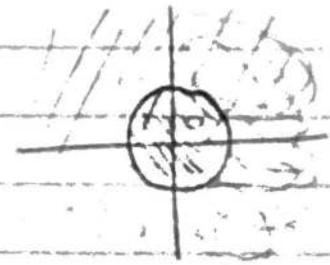
1. Compute critical points of f within K
2. Compute the max and min of f on those (CP)s
3. Optimize along the boundary curve

The max / min values are global extreme values of f on K

Ex: find global extrema of $f(x,y) = xy^2$ on $K = \{(x,y) : 0 \le x ; 0 \le y ; x^2+y^2 \le 3\}$

Sol: First compute critical points

Picture



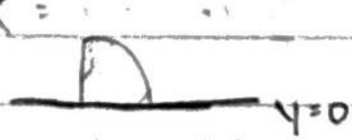
$\nabla f = \langle y^2, 2xy \rangle$

$\nabla f = \vec{0}$ iff $\langle y^2, 2xy \rangle = \vec{0}$

iff $\begin{cases} y^2 = 0 \\ 2xy = 0 \end{cases}$ iff $\begin{cases} y = 0 \\ x = 0 \ \& \ y = 0 \end{cases}$

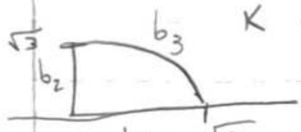
so iff $y = 0$

So K =



Note: in this example, boundary points contain the (CP)s that intersect K ... so analyzing the boundary will also analyze the (CP)s (skipping step 2 & just compute step 3)

Now lets analyze the boundary



Paramatrize the Boundry curves like so!

$b_1(t) = (t, 0)$ on $0 \le t \le \sqrt{3}$

$b_2(t) = (0, t)$ on $0 \le t \le \sqrt{3}$

$b_3(t) = (\sqrt{3}\cos t, \sqrt{3}\sin t)$ $0 \le t \le \frac{\pi}{2}$

Next we optimize $f(b_i(t))$ for each Boundry piece

On $b_1(t)$: $f(b_1(t)) = f(t,0) = t \cdot 0^2 = 0$ Both the Abs Max and abs min in B_1 & B_2

on $B_2(t)$: $f(b_2(t)) = f(0,t) = 0 \cdot t^2 = 0$

On $B_3(t)$: $f(b_3(t)) = f(\sqrt{3}\cos t, \sqrt{3}\sin t)$

$= \sqrt{3}\cos t \cdot \sqrt{3}\sin t^2$

$= 3\sqrt{3}\cos t \sin^2 t$

$= g(t)$

So $g'(t) = 3\sqrt{3}\left((-\sin t)\sin^2 t + \cos t(2\sin t \cos t)\right) =$

$g't = 3\sqrt{3}\sin t\,(2\cos^2 t + \sin^2 t)$

$\therefore g'(t) = 0$ iff $\sin(t) = 0$ or $2\cos^2(t) - \sin^2(t) = 0$

$2\cos^2 t = \sin^2 t$

Sin t & Cos t cannot both be 0 for the same value of t

Divide by $\cos^2 t$

Iff $\sin t = 0$. Or, $2 = \tan^2 t$

Iff $\sin(t) = 0$ or $\tan(t) = \pm\sqrt{2}$

Iff : $t = K\pi$ or $t = \operatorname{atan}(\sqrt{2})$ or $t = \operatorname{atan}(-\sqrt{2})$

for some integer K

$\to$

$0 \le t \le \frac{\pi}{2}$

iff $t = K\pi$ or $t = \operatorname{atan}(\sqrt{2})$ or $t = \operatorname{atan}(-\sqrt{2})$

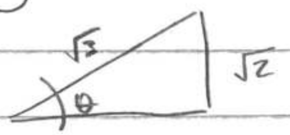
iff $t = 0$ or $t = \operatorname{atan}(\sqrt{2})$

$\cancel{\operatorname{atan}(-\sqrt{2})} \le 0$ So Reject

$\therefore$ testing $g(t)$ at the Boundary / Crit. pt. :

$g(0) = 0$, $g(\frac{\pi}{2}) = 0$

$g(\arctan(\sqrt{2})) = 3\sqrt{3}\cos(\operatorname{atan}(\sqrt{2}))\sin^2(\operatorname{atan}(\sqrt{2})) > 0$



$\theta = \operatorname{atan}(\sqrt{2})$

$\tan\theta = \frac{\sqrt{2}}{1}$

$\sin\theta = \frac{\sqrt{2}}{\sqrt{3}}$

$\cos\theta = \frac{1}{\sqrt{3}}$

$= 3\sqrt{3}\left(\frac{1}{\sqrt{3}}\right)\left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2$

$= 2 \longrightarrow 2 > 0$

$\therefore$ ABS max on K is 2 and the abs min on K is 0 for f $\boxed{}$

Q? : How Do we make Analogs for the 1^{st} & 2^{nd} Derivative test in Calc 3?

First Derivative test ! Let f be differentiable at Cp $\vec{p}$

① if $D_{\vec{u}} f(\vec{p} + \varepsilon\vec{u}) > 0$ for All Sufficent Small $\varepsilon > 0$ and all unit vectors $\vec{u}$, then f has a local min @ $\vec{p}$

② If $D_{\vec{u}} f(\vec{p} + \varepsilon\vec{u}) < 0$ for all Sufficent small $\varepsilon > 0$ and all unit vectors $\vec{u}$, then f has a local Max @ $\vec{p}$

$\rightarrow$

← NB: this is too hard to apply in this class
for problems ... Is there Anything Better?
yes, but there are failing conditions
To get a Second derivative test we need

$$D = \det \begin{vmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \end{vmatrix}$$

$$= f_{xx} \cdot f_{yy} - f_{xy} \cdot f_{yx}$$

$$= f_{xx} \cdot f_{yy} - (f_{xy})^2$$

takes place of the second derivative

Second derivative test: This only works as
Stated for $f(x,y)$ (2 variables)
Let $F(x,y)$ be differentiable at cp $(\vec{p})$

① if $f_{xx}(\vec{p}) > 0$ and $D(\vec{p}) = f_{xx}(\vec{p}) \cdot f_{yy}(\vec{p}) - (f_{xy}(\vec{p}))^2 > 0$
then $\vec{p}$ is a local min pt. of f

② If $f_{xx}(\vec{p}) < 0$ and $D(\vec{p}) = f_{xx}(\vec{p}) \cdot f_{yy}(\vec{p}) - (f_{xy}(\vec{p}))^2 > 0$
then $\vec{p}$ is a local max pt. of f

③ If $D(\vec{p}) = f_{xx}(\vec{p}) \cdot f_{yy}(\vec{p}) - (f_{xy}(\vec{p}))^2 < 0$, then
$\vec{p}$ is a Saddle point of f
(locally, f looks like a hyperbolic paraboloid at $\vec{p}$)